The question that I want to understand is: is it possible to define boundary conditions on a sphere for the maxwell equations such that the solution does not radiate and exist?

Take $c = 1$, Maxwel equation with source terms are:

$$\begin{aligned} \Box\phi &= -\delta_{r=r_n}\rho(e)/\epsilon_0 \\ \Box A &= -\delta_{r=r_n}\mu_0 j \\ \frac{\partial\phi}{\partial t}+\frac{\partial A_i}{\partial x_i} &= 0 \end{aligned}$$

Consider a solution at a hormonc frequency w_0. Solving this in the Fouruirer space we have,

$$\begin{aligned} \hat{\phi} &= \frac{1}{w_0^2-|s|^2}\int \rho/\epsilon_0 \exp(-i r_n e\cdot s) dS(e) \\ \hat{A} &= \frac{1}{w_0^2-|s|^2}\int \mu_0 j \exp(-i r_n e\cdot s) dS(e) \end{aligned}$$

The continuirt equation can be simplified to,

$$\begin{aligned} 0 &= \frac{1}{w_0^2-|s|^2}\int (i w_0 \rho/\epsilon_0 + i\mu_0(s\cdot j))\exp(-i r_n e\cdot s) dS(e) \\ &= \int (w_0\rho + (s\cdot j))\exp(-i r_n e\cdot s) dS(e) \end{aligned}$$

In order ot see verify the existance of a solution we need this to be true for all s, but that is the same as,

$$\lim_{s\to 0}\left(\prod_{k=1\ldots n}\frac{\partial}{\partial s_{i_k}}\right)\int (w_0\rho + (s\cdot j))\exp(-i r_n e\cdot s) dS(e) = 0$$

Doing the actual calulations yield the condition:

$$\begin{aligned} \int\left(w_0\rho(-ir)^n + \left(\sum_k j_{i_k}/e_{i_k}\right)(-ir)^{n-1}\right)\prod_k e_{i_k} dS(e) &= 0 \\ \int\left(w_0\rho + \left(\sum_k j_{i_k}/e_{i_k}/(-ir)\right)\right)\prod_k e_{i_k} dS(e) &= 0 \end{aligned}$$

So we need to find a differential so the expression inside the integral is d(Q(e)), then the integral becomes zero, note:

$$\nabla\cdot\left(\vec{a}\prod_k e_{i_k}\right) = \nabla\cdot\vec{a}\prod_k e_{i_k} + \sum_k \vec{a}\cdot\nabla(e_{i_k})/e_{i_k}\prod_k e_{i_k}$$

$$\begin{aligned} \nabla\cdot\vec{a} &= w_0\rho \\ \vec{a}\cdot\nabla(e_{i_k}) &= j_{i_k/}(-ir) \\ j_{i_k} &= (-ir)\vec{a}\cdot\nabla(e_{i_k}) \end{aligned}$$

Note that the zero of the integral follows from divergence theorem with the space as the spherical shell. This means that we must have

$$\begin{aligned} \int\rho \mathrm{dS} &= c\int d(a)\mathrm{dS} \\ &= 0 \end{aligned}$$

For spherical harmonics we have (because they are eigenvalues to the laplacian)

$$Y = c\nabla\cdot(\nabla Y)$$

and from divergence theorem we get

$$\begin{aligned}\int_S Y\,\mathrm{dS} &= \int_S c\nabla\cdot(\nabla Y)\\ &= \int_{\partial S} c\nabla Y\\ &= 0\end{aligned}$$

Because the sphere has no limit. Therefore it is expected to be able to find a from a spherical charge distribution. Now Assume that ρ is a spherical harmonics, then,

$$\nabla\cdot\vec{a} = w_0\,Y_{l,m}$$

If we equate $w_0 = c(l,m)$, then

$$\vec{a} = \nabla_S Y_{l,m}$$

whith this we can give a formule for the current,

$$j_i = (-ir)\overrightarrow{(\nabla_S Y_{l,m})}\cdot(\nabla_S(e_{i_k}))$$

A condition that is not satisfied for a constant density, for this we assume that ϕ does not depend in time and hence that term dissappears, similarly we then have

$$\int\sum_k j_{i_k}/e_{i_k}\prod_k e_{i_k}dS(e) = 0$$

now we can use,

$$\nabla_S\cdot\hat{\theta}\left(\prod_k e_{i_k}\right) = \sum_k \nabla_S\cdot\left(e_{i_k}\hat{\theta}\right)/e_{i_k}\prod_k e_{i_k}$$

And we can equate and solve for e_{i_k}, as:

$$j_i = \nabla_S\cdot\left(e_i\hat{\theta}\right)$$

which is doable, hence a solution exists. A good question is if the j vector points outward or not out of the spherical shell. Note,

$$\sum_i e_i j_i = \nabla_S\cdot\left(\sum e_i^2\right) = 0$$

So for the constant density j forms a continous vector field onto the sphere. For the case with spherical harmonics for the charge density a similar result,

We can similarly assume that the current is only defined through a circle in the xy-plane (coordinate 1 and 2) then similarly we have, Then $e_3 = 0$, and we get again the condition:

$$\begin{aligned}j_1 &= \frac{\partial}{\partial\varphi}(e_1) = -\sin(\varphi)\\ j_2 &= \frac{\partial}{\partial\varphi}(e_2) = \cos(\varphi)\end{aligned}$$

Indeed, because the integral is just $[\prod_k e_{i_k}]$ is defined on the circle the edges which are connected have the same value and the integral is zero. Mills is constructing the ground state through an integral of such loops and due to the superposition property there is no radiation. Now a good question is if the spherical harmonics soloutions densities will radiate or not. Mills proof of this is wrong due to a fault using convoluted fourier transforms.

Also please note that if we let I be the fourier transforms of the current sources then we have

$$\nabla \cdot I = 0$$

This specifically means that on the speed of light sphere S_{w_0}, we have,

$$\int_{S_{w_0}} I \cdot \hat{n}\, dS = 0$$

Not sure if this implies non radiation but's in line with it. What we can do is to evaluate:

$$s \cdot I = A \int Y_{l,m} \exp(i r_n e \cdot s)$$

and use,

$$\exp(i s \cdot u) = \sum_{l,m} a_l j_l(|s| \|u\|) Y_{l,m}(u_S) Y^t_{l,m}(s_S)$$

$$\int Y_{l,m}(x) Y^t_{l',m'}(x) \mathrm{dS}(x) = a(l,m) \delta_{l=l',m=m'}$$

to get,

$$s \cdot I = A a_l a(l,m) j_l(|s| r_n) Y_{l,m}(s_S)$$

And now to use Haus theorem we consider light like wave vector e.g. $|s| = w_0$ and get the condition introducing the speed of light equal to c again,

$$|s \cdot I| \leq C j_l(w_o r_n / c)$$

For $Y_{0,0}$, $j_0(x) = \mathrm{sinc}(x)$, the constraint for non radiation is

$$w_0 r_n / c = \pi k, k = 1, 2, 3, ...$$

For general l, we know that there are zeros $f_{l,k}$ and the condition is again,

$$w_0 r_n / c = f_{l,k}, k = 1, 2, 3, ...$$

All following through Mills quite well. The case with constant charge can be consider more in detail. We have, for v, such that,

$$\nabla_S \cdot v = w_0 \rho$$

That we can take the currents as,

$$j_i = v \cdot \nabla_S(e_i)$$

Using,

$$\nabla_S \cdot F = \frac{1}{r_n^2 \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) F_\theta) + \frac{1}{r_n^2 \sin(\theta)} \frac{\partial}{\partial \varphi} F_\varphi$$

Now for ρ constant, we can take

$$v = -w_0 \rho_0 r_n^2 \frac{\cos(\theta)}{\sin(\theta)} \hat{\theta}$$

Also,

$$\nabla_S f = \hat{\theta} \frac{1}{r_n} \frac{\partial f}{\partial \theta} + \hat{\varphi} \frac{1}{r_n \sin(\theta)} \cdot \frac{\partial f}{\partial \varphi}$$

This means

$$\begin{aligned} j_1 &= w_0 \rho_0 r_n \frac{\cos(\theta)}{\sin(\theta)} \cos(\varphi) \sin(\theta) \\ &= w_0 \rho_0 r_n \cos(\theta) \cos(\varphi) \end{aligned}$$

$$\begin{aligned} j_2 &= w_0 \rho_0 r_n \cos(\theta) \sin(\varphi) \\ j_3 &= -w_0 \rho_0 r_n \frac{\cos^2(\theta)}{\sin(\theta)} \end{aligned}$$

First of all we can verify that $e \cdot j(e) = 0$. Now,

$$\int_S j_1 \exp(i r_n s \cdot e) = c_1 \frac{\partial}{\partial s_x} \int \exp(i r_n s \cdot e) = c_1 \frac{\partial}{\partial s_x} \operatorname{sinc}(|s| r_n)$$

And

$$\int_S j_1 \exp(i r_n s \cdot e) \mathrm{dS} = c_1 \frac{\partial}{\partial s_y} \operatorname{sinc}(|s| r_n)$$

Now, for legendre polynomials we have

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \end{aligned}$$

Therefore

$$\frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) = x^2$$

And hence,

$$\begin{aligned} \frac{\partial}{\partial s_z} \int_S j_3 \exp(i r_n s \cdot e) dS &= i r_n \int_S \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) \exp(i r_n s \cdot e) dS \\ &= A j_3(|s| r_n) Y_3(s_S) + B \operatorname{sinc}(|s| r_n) \end{aligned}$$

Wave Packets

A plane wave solution to Maxwell's equatoin can be written as

$$\begin{aligned} \phi &= \phi_0 \exp(i w t + w e \cdot x) \\ A &= \phi_0 (e \times v - e) \exp(i w t + w e \cdot x) \end{aligned}$$

Using the expansion of the plane wave in spherical harmonics and multilpying with the weight $Y_{l,m}(\mathrm{e})$ and take the sphere integration yields,

$$\begin{aligned}\phi_{k,l} &= \phi_0 a_l\, j_l(w|x|)Y_{l,m}(\check{x})\mathrm{exp}(iwt)\\ A &= \phi_0 a_l(\nabla\times v-\nabla)\, j_l(w|x|)Y_{l,m}(\check{x})\mathrm{exp}(iwt)\\ &= \phi_0 a_l w(x\times v-x)\, j_l'(w|x|)/|x|Y_{l,m}(\check{x})\mathrm{exp}(iwt)+\phi_0\mathrm{a}\mathrm{j}_l(w|x|)_l(\nabla\times v-\nabla)Y_{l,m}(\check{x})\mathrm{exp}(iwt)\end{aligned}$$

And we see that for the non radiation condition at the source term sphere $\phi_{k,l}$ equals zero and if the outer of the sphere shuled have zero eleectric field then then you need a source term of the form $\mathrm{c}\mathrm{Y}_{l,m}(\check{x})$ just as proclaimed by Mills. This motivates why you have those terms. Note that a similar argument can be held at $-w$ and the complex conjugate of $Y_{l,m}$ essentially meaning that you can take the real part e.g. $\mathrm{Re}(Y_{l,m}\exp{(iwt)})$ and the same conclusion holds. So indeed we see that a photon can be traped inside the sphere giving rise to the math in Mills GUTCP. Also the actual calulation of the radious depends on a force balance where the speed of the charge field gotten from the angualr frequencey w, now the zeros of the sinc function is $k\pi$, and if hence we essentially get kw as the allowed angular speed giving rise to essentially the radious a_0/k. Alsi because the derivative of the sinc function at the zero is $\cos{(wkr_n)}/kwr_n$ we get the $\frac{1}{\mathrm{k}}$ factor found in Mills GUTCP. Also please note that letting $w\to 0$, and scaling $v=v/w^3$, we see that A turns into the static.

$$A = C_1\,(x\times v-x)|r|$$

using $B=\nabla\times A$ we get,

$$B = C_2 v|r|$$

Which is the static field Mills deduces in GUTCP inside the sphere and yields a solution which is the base for his calulation of the Landau g-factor to an amazing precition which is a great motivation to why we should take this model as a probable model of the atom.